thermal instability develops is in principle the same as before: beginning with a certain critical pressure gradient, the heat source ceases to balance the sink. If fluid velocity is less dependent on temperature than the rate of heat absorption, then critical phenomena are not observed; in this case, a decrease in fluid temperature leads to an equivalent decrease in the capacity of the heat sink, thereby stabilizing the processes of heat transfer and motion.

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MODEL OF THE TAYLOR INSTABILITY

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UDC 532.526

1. The Taylor (or Rayleigh–Taylor) instability arises on the boundary between a heavy and a light liquid in the field of gravity, when the heavy liquid is above the light one. The Taylor instability has been studied actively both experimentally [1] and theoretically with analytical methods [2-5], and by numerical simulation of the development of the instability [6-8]. Interest in this phenomenon has increased in recent years in connection with problems of control of thermonuclear synthesis with inertial confinement of a plasma. It was shown in [9, 10] that the development of an instability in the surface layers of a target is a Taylor instability and can lead to the deterioration of the symmetry of compression and characteristics of the target. In calculating the development of the Taylor instability in such problems a simple model is useful, which should describe the growth of perturbations with acceptable accuracy for practical purposes. In the present paper, using physically justifiable assumptions about the form of the perturbation, we simplify the formation of the problem on the development of the Taylor instability and obtain equations which correctly describe both the linear and asymptotic stages of the Taylor instability.

2. The usual formulation of the Taylor instability problem is as follows [4]. In a uniform gravity field with acceleration g directed downward, a heavy liquid with density ρ occupies the space $z > \theta(x, t)$ and a light liquid with density ρ_* ($\rho_* < \rho$) occupies the space $z < \theta(x, t)$, where $\theta(x, t)$ is the vertical position of the surface between the heavy and light liquids, and z and x are the vertical and horizontal Cartesian coordinates. At the time $t = 0$, the initial perturbation of the dividing surface is specified as

$$\theta(x,\ t=0) = \theta_0 \cos kx \tag{2.1}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 36-41, November-December, 1985. Original article submitted August 10, 1984.

0021-8944/85/2606-0788$09.50

with wavelength $\lambda = 2\pi/k$. It is required to find the subsequent evolution of the surface $\theta(x, t)$.

The periodicity of the initial perturbation (2.1), and its eveness with respect to x means that we can study the flow in the restricted interval $0 \leq x \leq \lambda/2$. It is known [4] that in the nonlinear stage of the instability, the initial sinusoidal form of the boundary is distorted: the heavy liquid penetrates into the light liquid in the form of thin "streamers," and the light liquid floats up into the heavy liquid in the form of "bubbles." In the region $z = 0$, $0 \leq x \leq \lambda/2$, the "bubble" occupies a cross section $S_+ = [0, x_0]$, and the "streamers" has a cross section $S_- = [x_0, \lambda/2]$, where x_0 is the "bubble"—"streamer" boundary. For the perturbation (2.1), the horizontal coordinate of the center of the "bubble" is $x = 0$; the corresponding coordinate of the center of the "streamer" is $x = \lambda/2$ and $x_0 = \lambda/4$. We will call the maximum penetration depth of the heavy liquid into the light one, measured from the equilibrium position $z = 0$, the amplitude of the "streamer," and the corresponding quantity for the light liquid will be called the amplitude of the "bubble." We attempt to construct a model of the Taylor instability which will describe the growth of perturbations in terms of the "streamer" and "bubble" amplitudes, both in the linear and nonlinear stages of the instability.

We consider the case $\rho_* = 0$. We assume potential flow of an incompressible, inviscid liquid. Then in the field of gravity g the Cauchy—Lagrange integral of the heavy liquid can be written in the form

$$\frac{\partial\Phi}{\partial t} + \frac{v^2}{2} + gz + \frac{p}{\rho} = \frac{v_0^2}{2} + \frac{p_0}{\rho}, \quad z > \theta(x, t), \tag{2.2}$$

where Φ is the velocity potential, defined in terms of the velocity field $\mathbf{v}$ by the relation $\mathbf{v} = \nabla\Phi$, $p(x, z, t)$ is the pressure field, $p_0 \equiv p(z \leq \theta)$, $v_0^2 \equiv v^2$ $(z = 0,\ x = x_0)$, ρ is the density of the heavy fluid, g is the gravitational acceleration, $\theta(x, t)$ is the vertical position of the perturbed surface, and x_0 is the "bubble"—"streamer" boundary.

We assume that the cross sections of the "bubble" S_+ and the "streamer" S_- are slowly varying in time, and then average (2.2) over the "bubble" and "streamer" cross sections for $z > \theta_{max}$:

$$\frac{\partial}{\partial t}\langle\Phi\rangle_\pm + \frac{1}{2}\langle v^2\rangle_\pm + gz + \frac{1}{\rho}\langle p\rangle_\pm = \frac{v_0^2}{2} + \frac{p_0}{\rho}. \tag{2.3}$$

Here $\langle v^2\rangle_\pm = \frac{1}{S_\pm}\int\limits_{S_\pm} ds v^2, \ldots;\ \theta_{max}$ is the maximum value of $\theta(x, t)$ at time t, the plus sign denotes quantities averaged over the "bubble" cross section, and the minus sign denotes quantities averaged over the "streamer" cross section S_-.

In particular,

$$\langle v^2\rangle_+ = \frac{1}{x_0}\int\limits_0^{x_0} dx v^2(x, z, t),\quad \langle v^2\rangle_- = \frac{1}{(\lambda/2 - x_0)}\int\limits_{x_0}^{\lambda/2} dx v^2(x, z, t).$$

We introduce the parameter

$$\gamma_\pm = \frac{\langle v^2\rangle_\pm}{\langle v_z^2\rangle_\pm}$$

and in place of (2.3) we write

$$\frac{\partial}{\partial t}\langle\Phi\rangle_\pm + \frac{1}{2}\gamma_\pm\langle v_z^2\rangle_\pm + gz + \frac{1}{\rho}\langle p\rangle_\pm = \frac{v_0^2}{2} + \frac{p_0}{\rho}. \tag{2.4}$$

We make the following assumptions in obtaining the basic equations of the Taylor instability model:

1. The perturbation has a "rectangular" profile, characterized by the parameters b (the "bubble" amplitude), a (the "streamer" amplitude), and x_0, the halfwidth of the "bubble," related to a and b by the equation of continuity

$$bx_0 = a(\lambda/2 - x_0). \tag{2.5}$$

2. Equation (2.4) can be continued to the region $z < \theta_{max}$ using the functional dependence obtained for $z > \theta_{max}$.

3. The functions in (2.4) can be obtained by using the asymptotic representation Φ_* for the velocity potential, which using the periodicity in x and the condition that the perturbations go to zero when $z \to +\infty$, has the form

$$\Phi_* = -q(t)\,e^{-kz}\cos kx,\quad z > \theta_{\max}. \tag{2.6}$$

From (2.6) we obtain the velocity field:

$$v_z = qke^{-kz}\cos kx = -k\Phi_*,$$
$$v_x = qke^{-kz}\sin kx,\quad v^2 = (qk)^2 e^{-2kz}. \tag{2.7}$$

It follows from (2.7) and the definition of the parameter $\gamma_\pm$ that $\gamma_\pm$ do not depend on z, but are functions only of x_0:

$$\gamma_+ = 1 + \frac{\int_0^{x_0} dx\,\sin^2 kx}{\int_0^{x_0} dx\,\cos^2 kx} = 1 + \frac{1 - \frac{1}{2\pi\delta}\sin 2\pi\delta}{1 + \frac{1}{2\pi\delta}\sin 2\pi\delta},$$
$$\gamma_- = 1 + \frac{\int_{x_0}^{\lambda/2} dx\,\sin^2 kx}{\int_{x_0}^{\lambda/2} dx\,\cos^2 kx} = 1 + \frac{1 + \frac{1}{2\pi(1-\delta)}\sin 2\pi\delta}{1 - \frac{1}{2\pi(1-\delta)}\sin 2\pi\delta},$$
$$\delta = \frac{x_0}{\lambda/2},\quad 1/2 \leqslant \delta < 1; \tag{2.8}$$

and in the limit $\delta \to 1$ $\gamma_+ \to 2$, $\gamma_- \to 1$.

Using the relation $\langle\Phi_*\rangle_\pm = -(1/k)\langle v_z\rangle_\pm$, which follows from (2.7), we have in place of (2.4)

$$-\frac{1}{k}\frac{\partial}{\partial t}\langle v_z\rangle_\pm + \frac{1}{2}\gamma_\pm\langle v_z^2\rangle_\pm + gz + \frac{1}{\rho}\langle p\rangle_\pm = \frac{v_0^2}{2} + \frac{p_0}{\rho}. \tag{2.9}$$

Then the equation for the "bubble" amplitude b is obtained from (2.9), putting $z = b$, $\langle v_z\rangle_+ = \dot b$, $\langle v_z^2\rangle_+ = \dot b^2$, $\langle p\rangle_+ = p_0$:

$$\ddot b = gkb - \frac{k}{2}\left[v_0^2 - \gamma_+\dot b^2\right]. \tag{2.10}$$

Similarly we can obtain an equation for the "streamer" amplitude a, putting $z = -a$, $\langle v_z\rangle_- = -\dot a$, $\langle v_z^2\rangle_- = \dot a^2$, $\langle p\rangle_- = p_0$ in (2.9):

$$\ddot a = gka + \frac{k}{2}\left[v_0^2 - \gamma_-\dot a^2\right]. \tag{2.11}$$

In order to close the system of equations (2.10) and (2.11), we must determine the quantity v_0^2. In general, this cannot be done. However note that if we omit the expressions in square brackets in (2.10) and (2.11), we obtain the usual equations of the linear theory for the growth of the perturbations. This means that we can interpret the terms inside the square brackets in (2.10) and (2.11) as corrections to the linear theory. Since for small perturbation amplitudes the representation of the velocity potential in the form (2.6) is correct also for $z < \theta_{\max}$, and the corresponding quantity v^2 depends only on z:

$$v^2 \sim e^{-2kz},$$

we use linear extrapolation on $v^2(z)$ from z = b to z = 0:

$$v_0^2 = \langle v^2(z=b)\rangle_+(1+2kb) = \gamma_+\dot b^2(1+2kb).$$

Then in place of (2.10) we have

$$\ddot b = gkb - \gamma_+ k^2 b\dot b^2. \tag{2.12}$$

Similarly for the expression in square brackets in (2.11), we represent v_0^2 in the form

$$v_0^2 = \langle v^2(z=0)\rangle_- = \langle v_x^2(z=0)\rangle_- + \langle v_z^2(z=0)\rangle_-.$$

Using the approximation

$$\langle v_x^2(z=0)\rangle_- = \langle v_x^2(z=b)\rangle_+ = (\gamma_+ - 1)\dot{b}^2,$$

$$\langle v_z^2(z=-a)\rangle_- \equiv \dot{a}^2 = \langle v_z^2(z=0)\rangle_-(1+2ka),$$

we obtain

$$[v_0^2 - \gamma_-\dot{a}^2] = (\gamma_+ - 1)\dot{b}^2 + \frac{\dot{a}^2}{1+2ka} - \gamma_-\dot{a}^2$$

and the equation for a in the form

$$\ddot{a} = gka + \frac{k}{2}\left[(\gamma_+ - 1)\dot{b}^2 + \frac{\dot{a}^2}{1+2ka} - \gamma_-\dot{a}^2\right]. \tag{2.13}$$

Analysis of Eqs. (2.12) and (2.13) shows that 1) for small perturbation amplitudes and velocities, (2.12) and (2.13) are nearly the same as the equations of the linear theory; 2) as the perturbation amplitude (and velocity) increases, the acceleration of the "bubble" decreases and approaches zero, such that asymptotically the "bubble" rises with a constant velocity; 3) with increasing perturbation amplitude (and velocity), the motion of the "streamer" approaches free fall in the field of gravity with acceleration g.

The asymptotic velocity of the "bubble" follows from (2.12)

$$v_* = \sqrt{g/2k} = 0.28\sqrt{g\lambda}.$$

This result agrees closely with the data of [1, 3-7]:

$$v_* = (0.2 - 0.3)\sqrt{g\lambda}.$$

We note that for large initial perturbation velocities v_{00}, the "bubble" experiences a strong braking. This leads to a "rapid" distortion of the perturbation profile, i.e., to a rapid change of x_0 in time. In this case the linear theory is inapplicable even for small perturbation amplitudes. It follows from (2.12) that the condition for a "rapid" variation of the profile is $\gamma_+ k^2\dot{b}^2 b \gg gkb$ or for $\gamma_+ = 2$, $\dot{b} = v_{00}$

$$2kv_{00}^2/g \gg 1.$$

One can estimate the growth of the perturbations in this case if in (2.13) we put

$$(\gamma_+ - 1)\dot{b}^2 = 2v_{00}^2 - \dot{b}^2, \ \gamma_- = 1.$$

Equations (2.12) and (2.13) can be generalized to the case where the density of the light liquid ρ_* is nonzero. The asymptotic potential in the light liquid is written in the form

$$\Phi_* = \xi(t)e^{kz}\cos kx, \ z < \theta_{\min},$$

where θ_{min} is the minimum value of the surface $\theta(x, t)$.

Averaging the Cauchy–Lagrange integral (2.2) for the above potential over the "bubble" and "streamer" cross sections, we obtain relations analogous to (2.9). Further, if we assume $\rho_* \ll \rho$, and ignore the square of the velocity of the light liquid and use the continuity conditions on the boundary of the light and heavy liquid for the quantities $\langle v_z\rangle_\pm$ and $\langle p\rangle_\pm$, we obtain the following equations for b and a:

$$\ddot{b} = g\left(\frac{\rho-\rho_*}{\rho+\rho_*}\right)kb - \gamma_+\left(\frac{\rho}{\rho+\rho_*}\right)k^2\dot{b}^2 b; \tag{2.14}$$

$$\ddot{a} = g\left(\frac{\rho-\rho_*}{\rho+\rho_*}\right)ka + \frac{k}{2}\left(\frac{\rho}{\rho+\rho_*}\right)\left[(\gamma_+ - 1)\dot{b}^2 + \frac{\dot{a}^2}{1+2ka} - \gamma_-\dot{a}^2\right]. \tag{2.15}$$

Equations (2.14) and (2.15) are consistent with (2.5) and (2.8) and are the basic equations of the Taylor instability model. For small perturbation amplitudes they agree with the linear theory, and for large amplitudes they give the correct asymptotic forms for the "bubble" and "streamer". It is not difficult to obtain the widths of the "bubble" and "streamer" using (2.5) and knowing the perturbation amplitude.

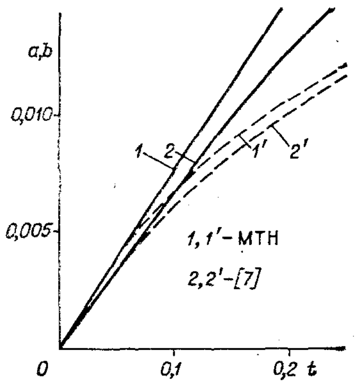

Fig. 1

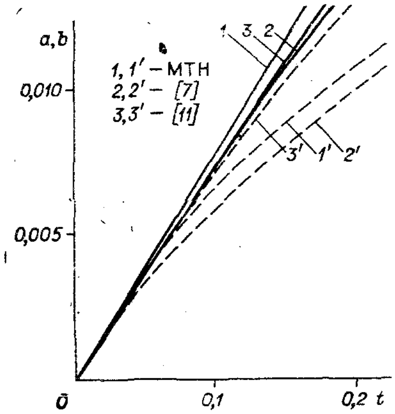

Fig. 2

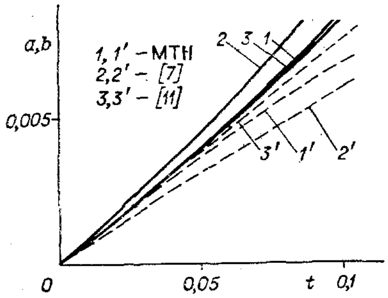

Fig. 3

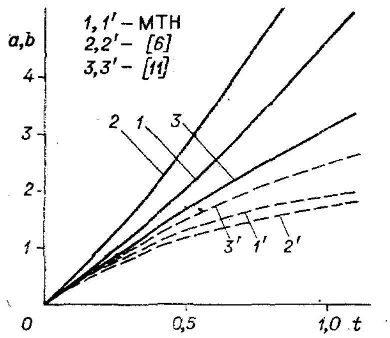

Fig. 4

3. The results for the Taylor instability model were compared with the corresponding two-dimensional calculations [7] for the values $\rho/\rho_* = 1.5$, 2.0, and 10 and are shown in Figs. 1-3; our results are compared with the two-dimensional calculations of [6] for $\rho = 1$, $\rho_* = 0$, and are shown in Fig. 4. In [7] the following perturbation parameters are used: $k = 157$, $a_0 = 0$, $v_{00} = 0.078$, $g = 1$, where a_0 is the initial amplitude of the perturbation. The parameter $2kv_{00}^2/g = 1.91$, which shows the effect of the nonzero density of the light liquid on the "slow" distortion of the profile, and consequently supports the applicability of equations (2.14) and (2.15) in this case. In [6] the following parameters were used: $k = 0.6545$, $a_0 = 0$, $v_{00} = 3.272$, $g = 1$. Then $2k_{00}^2/g = 14$, which corresponds to a "rapid" distortion of the profile, therefore in the equation for the "streamer" (2.15) we used $(\gamma_+ - 1)\dot{b}^2 = 2v_{00}^2 - \dot{b}^2$, $\gamma_- = 1$.

Figures 1 through 4 show the dependence of the "streamer" amplitude (solid curve) and "bubble" amplitude (dashed curve) on time. Comparison of the perturbation amplitudes calculated according to our model with the corresponding quantities obtained in the two-dimensional calculations of [6, 7] shows that the agreement is satisfactory. In Figs. 2 through 4 we also show the results for the heuristic model of the Taylor instability of [11]; overall the results of this model agree somewhat more poorly with the calculations of [6, 7].

The Taylor instability model given here does not contain phenomenological constants (unlike [11]) and can be used to estimate the growth of perturbations for the cases of both "slow" and "rapid" distortion of the perturbation profile for a wide range of ρ/ρ_*.

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